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And
Calculability of Observables

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Abstract

We reconsider quantum mechanical systems based on the classical action being the period of a one form over a cycle and elucidate three main points. First we show that the prepotential V is no longer completely arbitrary but obeys a consistency integral equation. That is the one form dV defines the same period as the classical action. We then apply this to the case of the punctured plane for which the prepotential is of the form $V = \alpha\theta + \Phi(\theta)$. The function Φ is any but a periodic function of the polar angle. For the topological information to be preserved, we further require that Φ be even. Second we point out the existence of a hidden scale which comes from the regularization of the infrared behaviour of the solutions. This will then be used to eliminate certain invariants preselected on dimensional counting grounds. Then provided we discard nonperiodic solutions as being non physical we compute the expectation values of the BRST- exact observables with the general form of the prepotential using only the orthonormality of the solutions (periodic). Third we give topological interpretations of the invariants in terms of the topological invariants which live naturally on the punctured plane as the winding number and the fundamental group of homotopy, but this requires a prior twisting of the homotopy structure

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1 Introduction

The interest on studying cohomological quantum mechanical systems here is to understand as a first step ,the way the supersymmetry and the ghost number are broken [1] in topological field theories, and the way to compute expectation values of the inherent topological invariants.In the class of topological field theories which can be rephrased as gauge fixed topological actions [2] [1] [4],observables are shown to be BRST- exact and have non vanishing expectation values ,but that presupposes that the underlying symmetry is somehow broken in a special way .To investigate these systems from the above point of view several difficulties arise.First the gauge fixing procedure through the use of a priori arbitrary prepotential V may spread out the initial topological information contained in the classical action.Second a hidden scale is shown to enter the analysis through the infrared regularization of the wavefunctions and this makes dimensional counting necessary but rather insufficient to preselect topological invariants.Third the identification of the observables with natural topological invariants needs first a kind of twisting of the homology structure.In this paper we investigate and elucidate these points.The starting point is the cancellation of the classical action during the process of the gauge fixing .This is a necessary requirement for the gauge fixed action to possess a secondary (dual) BRST symmetry,otherwise one cannot rewrite the hamiltonian in the form $H \propto \{Q, \bar{Q}\}$ with Q and \bar{Q} both nilpotent .The prepotential is then required to belong to the class of functions such that the one form dV defines the same period as the classical action in order to properly perform the above cancellation.In other words we simply require that for two quantities to identically cancel each other they should first be of the same (cohomological) nature.We then apply the idea to the case of the punctured plane and show that with the above selection criteria there is no loss or spread of the topological information as we could define things unambiguously.To eventually capture the right topological information without spurious backgrounds,the nonlinear term of the the prepotential in addition to its being periodic is further required to be an even function of the argument.The evenness of the prepotential is an essential input for the proper identification of the observables and will be justified both topologically and analytically in subsequent sections.On the other hand the problem is highly non trivial as it involves general Sturm-Liouville systems.We will show however that as long as we discard nonperiodic solutions which are seen not to reproduce orthonormality ,full knowledge of the spectrum and the associated eigenfunctions is hopefully not needed.The orthonormality of the periodic solutions only does really matter as a consequence of the even parity of the potential. The hidden scale is shown to come from Bessel functions which are known to be normalized only on finite "volume" ,hence a lenth L is introduced .This new scale will subsequently be used to discard certain "topological" invariants preselected on dimensional counting grounds.Finally the identification of the invariants in terms of the winding number W and the elements of the fundamental group of homotopy $\Pi^s(m)$ of the punctured plane calls for a type of twisting of the homotopy structure.The reason one has to do that is that the invariants to be identified being Q -dependant (Q -exact) depend explicitly on the coupling constants present in the theory as $Q \propto \partial V$ but these extra parameters (coupling constants) are not needed to characterize the (algebraic) topology of the punctured plane so one has to introduce them into the homotopy structure in one way or another.In an unpublished work [5] we performed such twisting of the homotopy structure by introducing an appropriate self interaction between homotopic loops viewed momentarily as intrinsec physical objects which may possibly self interact.In the process of identification and in the case of the two cases we studied we arrived at the following identifications of the invariants $W + i\alpha 1$ and $J + i\alpha 1$ with $J = -\sum_{m \in \mathbb{Z}/0} \rho(m) \frac{\Pi(m)}{m}$ where W is the winding number , 1 is the identity of the group of homotopy and the $\Pi^s(m)$ $m \neq 0$ are the rest of the group elements. $\rho(m)$ is an m -dependant coupling constant with had acquired a priori a meaning in the context of a twisted homotopy.

2 Gauge fixing topological actions

Let M be a compact manifold with local coordinates x^μ and let $H^r(M)$ be the r^{th} de Rham cohomology group. If c_1, \dots, c_k are elements of the homology group $H_r(M)$ with k the r^{th} Betti number such that $[c_i] \neq [c_j]$, then for any set of numbers b_1, \dots, b_k , a corollary of the de Rham's theorem states that there exist a closed r -form ω such that.

$$\int_{c_i} \omega = b_i \quad 1 \leq i \leq k \quad (1)$$

We may include in this expression the case $b_i = 0$ for which ω is closed and exact. The numbers in 1 are the periods of closed r -forms over cycles c_i . Cohomological quantum mechanics is defined as a system whose classical action is the period b_1 . The main feature of such an action is that it is defined on $H_r(M) \times H^r(M)$ and is therefore topological, that is invariant under any infinitesimal deformation ϵ which keeps the cycle within its homology class.

$$\delta x^\mu = \epsilon^\mu \quad (2)$$

The BRST symmetry associated to the above symmetry can be chosen as simple as.

$$\begin{aligned} s x^\mu &= \psi^\mu \\ s \psi^\mu &= 0 \\ s \bar{\psi}^\mu &= \lambda^\mu \\ s \lambda^\mu &= 0 \end{aligned} \quad (3)$$

Provided one chooses a quite involved gauge function. To gauge fix the symmetry such that covariance is maintained and to get an action quadratic in velocities Baulieu and Singer [3] proposed the following gauge function.

$$\dot{x}^\mu + \frac{\partial V}{\partial x^\mu} + \dots \quad (4)$$

Where $\dot{x}^\mu = \frac{dx^\mu}{dt}$ and where we omit the Christoffel symbol term as these are known matters and not relevant for what follows. We will concentrate in this section on the prepotential V . The prepotential is a priori an arbitrary given function of the coordinates x^μ . The expectation values of the invariants may depend on the form of the class of the functions V as the Euclidean path integral probes the moduli space of the equation $\dot{x}^\mu + \frac{\partial V}{\partial x^\mu}$. In this paper we want to show among other things that the prepotential is not that arbitrary but is restricted by a consistency equation. Write the gauge fixed action.

$$S_{SG} = \int_{c \in H_1} \omega + \int dt \ s \bar{\psi}^\mu (g_{\mu\nu} \dot{x}^\nu + \frac{\partial V}{\partial x^\mu} - \frac{1}{2} g_{\mu\nu} \lambda^\nu + \dots) \quad (5)$$

After integrating out the auxillary field λ we select out the bosonic linear term of interest

$$\int \dot{x}^\mu \frac{\partial V}{\partial x^\mu} dt = \int dV \quad (6)$$

The second step is to cancel out the classical action. This cancellation is a necessary requirement for the resulting gauge fixed action to possess a secondary (dual) BRST symmetry which in turn allows the hamiltonian to have the required form $H \propto \{Q, \bar{Q}\}$ with Q and \bar{Q} both nilpotent. Now comes our remark that in order for two quantities to identically cancel each other they should first be of the same

(cohomological) nature. Therefore the prepotential V should be such that the one form in 6 is a period and that this equates the classical action.

$$\int_{c \in H_1} \omega = \int_{c \in H_1} dV \quad (7)$$

In other words ,one should not just look for a prepotential on the basis that the integral of it numerically cancels out the classical action. We will check in the case of the punctured plane which we will study in details how restrictive the selection criteria is and how indeed it encompasses the entire topological information of the classical action. A completely arbitrary choice of the prepotential will eventually spread out the initial topological information. We hereafter specialize to the case of the punctured plane $R^2/(o)$ and take it as our target manifold, as it has simple but not trivial topology. In this case we have .

$$H^1(R^2/(0), R) \cong R \quad (8)$$

The ω 's are then one forms labelled by real numbers and the cycles c are homotopic loops encircling the whole as $H_1 \cong \Pi_1 \cong Z$ where Π_1 is the first homotopy group or the fundamental group and Z is the set of integers. The topological action associated with the punctured plane is , $\alpha \in R$ [6]

$$S_{cl} = \int_c \alpha \, d\theta \quad (9)$$

Where we define polar coordinates as $x^1 + ix^2 = re^{i\theta}$. The solution to the equation 7 for the case of the punctured plane is.

$$V = \alpha\theta + \Phi(\theta) \quad (10)$$

The function $\Phi(\theta)$ is any but periodic function of θ . This form is entirely due to the cohomological nature of the punctured plane. The simplest case $\Phi = 0$ has been selected by Baulieu and Rabinovici [1] on the basis of local BRST. Let us open a parenthesis about the use of local BRST in the present context as the latter is considered the important input of reference [1]. Local BRST symmetry provided the authors with an arrow of selective differential equations for the prepotential which we rewrite in a form more appropriate for the discussion. (To be compared to our selection criteria 7)

$$\frac{\partial}{\partial x_j} (x_i \frac{\partial V}{\partial x_i}) = 0 \quad (11)$$

As it is written this equation simply means that the prepotential $V(r, \theta)$ is in fact function of the θ variable only and should not depend on the radial variable r as one may write $x_i \frac{\partial V}{\partial x_i} = r \frac{\partial V}{\partial r}$. We therefore conclude that local BRST invariance is poorly restrictive as any function of θ may well do and so it is not enough to select the particular value $\alpha \theta$ of the prepotential as the above authors did. In subsequent sections we will solve the problem completely using the full expression of the prepotential 10 including the Φ term.

3 Bessel functions and the definition of observables

In the following we will compute observables in the canonical formalism. The hamiltonian associated with the action 5, which we adapt to the punctured plane is .[1]

$$\begin{aligned} H &= \frac{1}{2} p^2 + \frac{1}{2r^2} \left(\frac{\partial V}{\partial \theta} \right)^2 - \frac{1}{2} [\bar{\psi}_i, \psi_j] \frac{\partial^2 V}{\partial x_i \partial x_j} \\ &= \frac{1}{2} \{Q, \bar{Q}\} \end{aligned} \quad (12)$$

Where $p_i = -i\frac{\partial}{\partial x_i}$ and $\bar{\psi}_i = \frac{\partial}{\partial \psi_i}$ are canonical momenta for the coordinates x_i and the ghost fields ψ_i respectively and where the generators Q and \bar{Q} are given by.

$$\begin{aligned} Q &= \psi_i(p_i + i\frac{\partial V}{\partial x_i}) \\ \bar{Q} &= \bar{\psi}_i(p_i - i\frac{\partial V}{\partial x_i}) \end{aligned} \quad (13)$$

Let us first consider the eigenvalue problem associated to the hamiltonian 12 without specifying for the moment the form of the prepotential. For this purpose it is appropriate to adopt the superwavefunction formalism. Let $\Phi(x, \psi)$ denotes the superwavefunction.

$$\Phi(x, \psi) = \phi + \psi_i A^i + \frac{1}{2} \epsilon_{ij} \psi_i \psi_j B \quad (14)$$

Where the four states ϕ, A^i and B are functions of the coordinates x_i only. On the superspace the hamiltonian is represented as.

$$H = -\frac{\partial^2}{\partial x_i \partial x_i} + \frac{1}{2r^2} \left(\frac{\partial V}{\partial \theta}\right)^2 - \frac{1}{2} \left[\frac{\partial}{\partial \psi_i}, \psi_j\right] \frac{\partial^2 V}{\partial x_i \partial x_j} \quad (15)$$

The eigenvalue equation on the superspace

$$H\Phi = E\Phi \quad (16)$$

is projected on each component and gives the set of eigenvalue equations.

$$\begin{aligned} HB &= EB \\ H\Phi &= E\Phi \\ (H\delta_{ij} + \frac{\partial^2 V}{\partial x_i \partial x_j})A^i &= EA^j \\ H &= -\frac{\partial^2}{\partial x_i \partial x_i} + \frac{1}{2r^2} \left[\left(\frac{\partial V}{\partial \theta}\right)^2 - \frac{\partial^2 V}{\partial^2 \theta}\right] \end{aligned} \quad (17)$$

All these equations are of the same type (the third one has to be diagonalized) ,we therefore restrict ourselves to the ghost zero wavefunction Φ for instance. Let us look for a solution of the form.

$$\mathcal{F}(r)\Psi(\theta) \quad (18)$$

Putting this into the Φ equation in 17 and separating the variables, we get.

$$\begin{aligned} \ddot{\Psi}(\theta) + (\zeta^2 - W)\Psi(\theta) &= 0 \\ \ddot{\mathcal{F}} + \frac{1}{2}\dot{\mathcal{F}} + \left(\frac{\zeta^2}{r^2} + 2E\right)\mathcal{F} &= 0 \\ W &= \left(\frac{\partial V}{\partial \theta}\right)^2 - \left(\frac{\partial^2 V}{\partial^2 \theta}\right) \end{aligned} \quad (19)$$

Where ζ is a separation parameter. As topological theories are based on the independence on scales (any) ,the second (Bessel) equation is relevant in this respect. We therefore should elucidate some important

aspects of the solutions which has been ignored or simply underestimated in reference [1]. The solutions to the Bessel equation in 19 are .

$$\mathcal{F} \propto J_\zeta(\sqrt{2}Er) \quad (20)$$

J_ζ is a Bessel function of order ζ . One of the properties of the Bessel function is that it vanishes asymptotically as the argument E goes to zero for fixed r . This means that the system has no admissible ground state. The hamiltonian being of the form $H \propto \{Q, \bar{Q}\}$, all accessible states ($E \neq 0$) are not Q, \bar{Q} invariant. The supersymmetry is therefore broken in a way that no mass gap is introduced. This in turn opens the possibility of having BRST-exact operators with non-vanishing expectation values, which in addition will be shown to be topological in the sense that they do not depend on any scale present in the theory. A second property which is very relevant to the definition of topological observables is that Bessel functions are known to be (ortho)normalizable only on a finite "volume". To define orthonormality we introduce an infrared cut-off. This is a new scale not present in the lagrangian. This length is defined such that (We omit the $\sqrt{2}$ factor for convenience in what follows)

$$J_\zeta(EL) = 0 \quad (21)$$

For a fixed L this equation has an infinite set of roots E_s . Bessel functions are thus normalized consequently as [9].

$$\int_0^L r J_\zeta^2(E_s r) dr = \frac{L^2}{2} J_{\zeta+1}^2(E_s L) \quad (22)$$

This new scale will serve to properly define topological invariants.

4 Calculability of observables and the Sturm-Liouville system

Let us now concentrate on the angular dependance of the solutions. This is given by the equation

$$\begin{aligned} \ddot{\Psi}(\theta) + (\zeta^2 - \alpha^2 - W)\Psi &= 0 \\ W &= \left(\frac{\partial \Phi}{\partial \theta}\right)^2 - \left(\frac{\partial^2 \Phi}{\partial^2 \theta}\right) \end{aligned} \quad (23)$$

In the above equation the prepotential has been replaced with the general solution 10. We will distinguish two different cases of interest which will be shown to lead to different topological informations. The first case of vanishing potential is trivial and corresponds to the potential $W = 0$

$$\ddot{\Psi}(\theta) + (\zeta^2 - \alpha^2)\Psi = 0 \quad (24)$$

The solutions are simply $\Psi = e^{in\theta}$ where $\zeta^2 = n^2 + \alpha^2, n \in \mathbb{Z}$. The second case of nonvanishing potential is highly non trivial. In this case the prepotential is quite general and is only required to be r -independent and periodic in θ as a consequence of the cohomology of the target manifold. We hereafter, as far as the solutions of the angular equation are concerned will understand ζ^2 as shifted by α^2 . Equation 23 with W periodic is known as Hill's equation [10]. We will show in the next section that if one makes the judicious choice of an even potential W , only the existence and the orthonormality of the solutions will be needed for the computation of the vacuum expectation values of the relevant topological invariants. Let us convert Hill's equation to the well understood Sturm-Liouville systems in which conditions for the existence and the orthonormality of the solutions are more transparent. To this end we use the periodicity property of the potential to write the solutions as Bloch wavefunctions. That is .

$$\Psi(\theta) = \exp(ik\theta)u_k(\theta) \quad 0 \leq k < 1 \quad (25)$$

Where k 's are the eigenvalues of the translation operator (periodicity) and where $u_k(\theta)$ are periodic functions with the same period as the potential W . Putting this into 23 we get .

$$\ddot{u}_k + 2iku_k + (\zeta^2 - k^2 - W)u_k = 0 \quad (26)$$

As any second order ,linear ,homogeneous differential equation,it can be transformed to the form.

$$\begin{aligned} Lu_k &= -\lambda s u_k \\ L &= \frac{d}{d\theta} \left(p(\theta) \frac{d}{d\theta} \right) + q(\theta) \end{aligned} \quad (27)$$

Where the $p(\theta), q(\theta)$ functions and λ are defined as follows.

$$\begin{aligned} s(\theta) &= e^{2ik\theta} \\ p(\theta) &= e^{2ik\theta} \\ q(\theta) &= -W e^{2ik\theta} \\ \lambda &= -k^2 + \zeta^2 \end{aligned} \quad (28)$$

This is a singular Sturm-Liouville system [11] with periodic boundary conditions .Any such system with periodic boundary conditions is called regular and have orthonormal solutions if the function $p(\theta)$ is continuously differentiable and

$$\begin{aligned} p(\theta) &\geq 0 \\ p(0) &= p(2\pi) \end{aligned} \quad (29)$$

The first condition fails as the function $p(\theta)$ is oscillatory and thus may take positive as well as negative values,the second ,because k is real .We will therefore limit ourselves to the set of orthonormal solutions and put $k=0$.This is the subset of periodic solutions. The nonperiodic solutions may anyhow be discarded on the basis that they probably do not describe Hilbert spaces due to their not being (orth)normal .In this subspace the Sturm-Liouville system get regular and simplify to the form.

$$\ddot{u} + (\zeta^2 - W)u = 0 \quad (30)$$

Note that this is of the same form as the starting equation in 23 but here the solutions are now periodic. There is a set of theorems [11] in the chapter of boundary and eigenvalue problems which state that the above equation independently of the potential W provided it is continuous ,has real eigenvalues and all ordered as (In writing ζ^2 in the equation above we have already anticipated the positivity of the eigenvalues)

$$\begin{aligned} 0 &< \zeta_1^2 < \dots < \zeta_p^2 \\ \lim_{p \rightarrow \infty} \zeta_p &= \infty \end{aligned} \quad (31)$$

And moreover that the corresponding eigenfunctions are orthonormal with weight 1 .As we said already the exact expressions for the eigenfunctions and values of the spectrum will not be relevant to our purpose. It is clear that the Φ term in the prepotential V can be chosen at will (even or odd) provided it is periodic. If Φ is even then so is the potential W . Φ and W have the same parity only if Φ is even. On the other hand the operator $\frac{\partial^2}{\partial \theta^2} + (\zeta^2 - W)$ in 30 commutes with the parity operator $\mathcal{P}u_\zeta(\theta) = u_\zeta(-\theta)$.The solutions u_ζ can then be chosen as eigenfunctions of \mathcal{P} as well, that is either even

or odd functions of θ . Being periodic they could be expanded on a general exponential basis. If $u_{\zeta p}$ belongs to the subset of even functions for instance then it can be expanded on the cosine basis only.

$$u_{\zeta p} = \sum_{m=0}^{\infty} A_m^{\zeta p} \cos m\theta \quad (32)$$

We are now ready to write down vacuum expectation values for topological observables. On dimensional grounds, one may select the following candidates, together with their hermitian conjugates.

$$\begin{aligned} \mathcal{O}_\theta &= \{Q, \epsilon^{ij} x_i \bar{\psi}_j\} = -i\partial_\theta + i\partial_\theta V \\ \mathcal{O}_r &= \{Q, i x_j \bar{\psi}_j\} = r\partial_r \end{aligned} \quad (33)$$

In writing these expressions we dropped ghost terms as they give vanishing actions on the Φ wavefunctions we are considering. Let us first show that the second candidate is discarded as it depends explicitly on energy.

$$\begin{aligned} \langle \mathcal{O}_r \rangle_{E, \zeta} &= \frac{\int_0^L J_\zeta \partial_r J_\zeta r^2 dr}{\int_0^L J_\zeta^2 r dr} \\ &= -1 + \frac{J_\zeta^2(E_s L)}{J_{\zeta+1}^2(E_s L)} \end{aligned} \quad (34)$$

This expectation value is indeed equal to -1 as we have $J_\zeta(E_s L) = 0$ by construction. It seems then that it depends neither on E nor on L , but this result is only true for a denumerable set of energies E_s . Any small deviation from these energies would lead to different values for the expectation value. To see this expand the right hand side of 34 around E_s . Put $E = E_s + \epsilon$.

$$\begin{aligned} J_\zeta(EL) &= J_\zeta(E_s L) + \epsilon L J'_\zeta(E_s L) \\ &= -\epsilon L J_{\zeta+1}(E_s L) \end{aligned} \quad (35)$$

Where we have used the relations $J_\zeta(E_s L) = 0$ and $J'_\zeta(E_s L) = -J_{\zeta+1}(E_s L)$. J' being the derivative of J . Putting this into 34 we get .

$$\langle \mathcal{O}_r \rangle_{E, \zeta} \approx -1 + \epsilon^2 L^2 \quad (36)$$

Which shows an explicit dependance on energy and on L . The other candidate turns out to be a topological invariant. The expectation value of \mathcal{O} (we drop the index) between orthonormal states $u_{\zeta p}$ is.

$$\langle \mathcal{O} \rangle_{E, \zeta} = \frac{\int u_\zeta^* (-i\partial_\theta + i\partial_\theta V) u_\zeta d\theta}{\int u_\zeta^* u_\zeta d\theta} \quad (37)$$

In the case of a vanishing potential for which the solutions are $u_n = e^{in\theta}$ we get

$$\langle \mathcal{O} \rangle_{E, n} = n + i\alpha \quad (38)$$

In the case of non vanishing potentials the orthogonality of the basis cancels the ∂_θ contribution as we have.

$$\int u_\zeta^* \partial_\theta u_\zeta d\theta = 0 \quad (39)$$

Only the c-number term in 37 will then contribute,hence .

$$\begin{aligned}
\langle \mathcal{O} \rangle_{E,\zeta} &= i\partial_\theta V \\
&= - \sum_{m \in \mathbb{Z}/(0)} \frac{\rho(m)}{m} e^{im\theta} + i\alpha 1
\end{aligned} \tag{40}$$

Where the "spectral" function $\rho(m)$ is even in m as V is even in θ . The potential being periodic we have resolved it into its Fourier components.It is an important fact that V may be selected with even parity as this forced the solutions to be of a given parity and thus to discard the unwanted contribution coming from the derivative operator (angular momenta).In the next section we will give a further justification of the selected parity for the prepotential as a necessary requirement for proper identification of the observables as topological invariants of the punctured plane.

5 Twisting of the homotopy structure and topological interpretation

Now to interpret the Q-exact invariants in terms of the invariants which live naturally on the punctured plane we should first recall some known results and modify the homotopy structure somehow.The topology of the punctured plane is encoded within the fundamental group of homotopy Π_1 which is isomorphic to the set of integers, $\Pi_1 \cong \mathbb{Z}$. Thus each topological state is labelled by the integers and will be denoted $|n\rangle$.We then have a set of operators at our disposal W the winding number and $\Pi_1(m)$ the elements of the homotopy group which act on them .The whole topological information may be summarized in the following set of equations.

$$\begin{aligned}
W |n\rangle &= n |n\rangle \\
\Pi_1(m) |n\rangle &= |n+m\rangle \\
\sum_{n \in \mathbb{Z}} |n\rangle \langle n| &= 1
\end{aligned} \tag{41}$$

The group Π_1 being abelian , all its elements are generated by $\Pi_1(0) = 1$ the identity and the generator $\Pi_1(1)$ for instance.

$$\Pi_1(m) = \Pi_1^m(1), \quad m \in \mathbb{Z} \tag{42}$$

To make the connection with Q-exact invariants we convert to the theta basis.The fundamental group being abelian ,its unitary irreducible representations are one dimensional.Write these states as $|\theta\rangle$ where θ is an angle.The operators Π_1^s and W act on these states as.

$$\begin{aligned}
\Pi_1(m) |\theta\rangle &= e^{im\theta} |\theta\rangle \\
W |\theta\rangle &= i\partial_\theta |\theta\rangle
\end{aligned} \tag{43}$$

On the other hand both kinds of states are related to each other through the Fourier transform.

$$\begin{aligned}
|\theta\rangle &= \sum_{m \in \mathbb{Z}} e^{-im\theta} |m\rangle \\
|m\rangle &= \int_0^{2\pi} e^{im\theta} |\theta\rangle \frac{d\theta}{2\pi}
\end{aligned} \tag{44}$$

At this stage one could not propose to identify the resulting topological invariants in terms of the homotopy of the target manifold. That is to identify in the case at hand the invariants in terms of the winding number W and the elements of the fundamental group of homotopy $\Pi_1^s(m)$. To make the identification properly one should perform a twisting of the homotopy structure. In an unpublished work [5] we performed such twisting. What we mean by that is that by viewing momentarily topological loops as intrinsic physical objects we allow them to have self interactions. The reason one has to do that is that the invariants to be identified being Q-exact depend explicitly on the coupling constants present in the theory as $Q \propto \partial V$ but these extra parameters (coupling constants) are not (present) needed to characterize the (algebraic) topology of the punctured plane. One should enrich the homotopy structure somehow to implement these parameters. We will just quote the results which could be checked without much effort. Starting from the operators W and $\Pi(m)$ (Hereafter we omit the subscript 1), we may build a new functional operator $W_\lambda[\rho]$ an " effective winding number " as .

$$\begin{aligned} W_\lambda[\rho] &= W + \lambda \sum_{m \in \mathbb{Z}} \rho(m) \Pi(m) \\ &= e^{\lambda J} W e^{-\lambda J} + \lambda \rho(0) \\ \text{With } J[\rho] &= - \sum_{m \in \mathbb{Z}/(0)} \rho(m) \frac{\Pi(m)}{m} \end{aligned} \quad (45)$$

To get the first equation from the second one ,one may use the commutation relation $[W, \Pi(m)] = m \Pi(m)$ one may extract from the set of defining equation 42. Let us first make clear the choice of the special interaction in 46 by looking at the second equation .The first term (set $\rho(0) = 0$) is just the twisting of the winding number operator W . This is the analog of the twisting of the exterior derivative operator first performed by Witten [12] in order to introduce the prepotential in a cohomological way let us say. In the above formula we did the same thing but in a homological way. (Recall that the isomorphism $H_1 \cong \Pi_1$ means that homology and homotopy are equivalent words in the present case). So one has to compare the above twisting with

$$d_\lambda[V] = e^{\lambda V} d e^{-\lambda V} \quad (46)$$

Where $d \equiv W = i\partial_\theta$ (on the $|\theta\rangle$ states). With this we see that we have done more than a twisting as we added an affine non-vanishing term $\lambda\rho(0)$. This term is fundamental as it permits to define not only a twisted version of the system but another physically different one. It is interesting to regard the operator $W_\lambda[\rho]$ as a hamiltonian which describe interacting homotopic loops with. W as the unperturbed part and where $\lambda\rho(m)$ is an m -dependant coupling constant function. In the above cited reference [5] we solved the eigenvalue problem associated to the above hamiltonian and find that it possesses a set of eigenstates all well defined and having real eigenvalues to which only the affine term contributes. $n_{\lambda\rho(0)} = n + \lambda\rho(0)$ with $n \in \mathbb{Z}$ and $0 < \lambda\rho < 1$. In the case of pure twisting $\rho(0) = 0$ the spectrum remains unchanged as we know that the twisting operation is a kind of symmetry for the system. The new eigenstates are related to the old ones through the formula .

$$|n, \lambda, \rho\rangle = e^{\lambda J} |n\rangle \quad (47)$$

These states describe interacting homotopic loops and may be interpreted as a new larger basis for the homotopy structure which reduces to the usual homotopy basis once we set $\lambda = 0$. But to get a topologically interesting system we further require that the hamiltonian $W_\lambda[\rho]$ be hermitian with the function $\rho(m)$ being real. This fixes the latter function to be an even function in m that is $\rho(-m) = \rho(m)$. As a consequence the operator $e^{\lambda J}$ is unitary (J antihermitian). Has this operator being not unitary which could arise if ρ is odd or having no parity the new states which will no longer be the

unitary transform of the old ones .This would certainly spread out the initial topological information about the homotopy structure available before we switch on the interaction.This observation should be seen as a further (here topological) justification of why we adopted periodic potentials of even parities which was translated on the ρ function in 40 being an even function of m . Recall that this condition was necessary and sufficient to get rid of the unwanted contributions coming from the angular momenta which would require full knowledge of the spectrum and the associated eigenfunctions. We may now give the topological interpretation .For the two cases the identifications are as follows. In the first case we get .

$$\mathcal{O} = W + i\alpha \, 1 \quad (48)$$

This is the effective winding number $W_\lambda[\rho]$ in 46 with the operator J set to zero and where $\lambda\rho(0) = \alpha$. We see in this case that there is no need to twist the winding number .For the second case for which the prepotential is periodic ,even and the nonperiodic solutions discarded on comparing the expectation value of the Q -exact invariant in equation 40 with that of the J operator taken between $|\theta\rangle$ states the identification is.

$$\mathcal{O} = J + i\alpha \, 1 \quad (49)$$

To end up we may quote another result from our recent work [13]on the topological setting of Bessel functions in order to give a further meaning to the invariant J .We have shown there that if one associates a reduced Bessel function $j_n(z) = \frac{J_n(z)}{z^n}$ to the loop state $|n\rangle$ then picking the particular value for the coupling constant $\rho(m) = (-)^m$ one may rewrite the formula 47 as .

$$j_{n+\lambda} = e^{\lambda J} j_n \quad (50)$$

Where we can see that the invariant J we got ,for the typical choice of the spectral function ρ is the generator of real order reduced Bessel functions (from integer order ones).

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